

Monty Hall, Monty Fall, Monty Crawl

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1 Introduction

In 1990, vos Savant [3] introduced the infamous *Monty Hall problem*. Her asserted answer set off a storm of controversy in which she received thousands of letters [4]. Numerous professional mathematicians and others insisted that she was wrong, some using rather strong language (“you are utterly incorrect”; “I am in shock”; “you are the goat”). Vos Savant had the last laugh, when she called upon “math classes all across the country” to estimate the probabilities using pennies and paper cups, and they reported with astonishment that vos Savant was correct [4].

Despite all the publicity, most people have at best a vague understanding of why vos Savant’s answer is correct, and the extent to which it does or does not also apply to variants of the problem. In this paper, we discuss the Proportionality Principle, which allows this and many related problems to be solved easily and confidently.

2 The Monty Hall Problem and Variants

The original Monty Hall problem may be summarised as follows:

Monty Hall Problem: A car is equally likely to be behind any one of three doors. You select one of the three doors (say, Door #1). The host then reveals one non-selected door (say, Door #3) which does *not* contain the car. At this point, you choose whether to stick with your original choice (i.e. Door #1), or switch to the remaining door (i.e. Door #2). What are the probabilities that you will win the car if you stick, versus if you switch?

Most people believe, upon first hearing this problem, that the car is equally likely to be behind either of the two unopened doors, so the probability of winning is $1/2$ regardless of

whether you stick or switch. However, in fact the probabilities of winning are $1/3$ if you stick, and $2/3$ if you switch [3]. This fact is often justified as follows:

Shaky Solution: When you first selected a door, you had a $1/3$ chance of being correct. You *knew* the host was going to open some other door which did not contain the car, so that doesn't change this probability. Hence, when all is said and done, there is a $1/3$ chance that your original selection was correct, and hence a $1/3$ chance that you will win by sticking. The remaining probability, $2/3$, is the chance you will win by switching.

This solution is actually correct, but I consider it “shaky” because it fails for slight variants of the problem. For example, consider the following:

Monty Fall Problem: In this variant, once you have selected one of the three doors, the host slips on a banana peel and *accidentally* pushes open another door, which just *happens* not to contain the car. *Now* what are the probabilities that you will win the car if you stick with your original selection, versus if you switch to the remaining door?

In this case, it is still true that originally there was just a $1/3$ chance that your original selection was correct. And yet, in the Monty Fall problem, the probabilities of winning if you stick or switch are both $1/2$, not $1/3$ and $2/3$. Why the difference? Why doesn't the Shaky Solution apply equally well to the Monty Fall problem?

Another variant is as follows:

Monty Crawl Problem: As in the original problem, once you have selected one of the three doors, the host then reveals one non-selected door which does not contain the car. However, the host is very tired, and *crawls* from his position (near Door #1) to the door he is to open. In particular, if he has a choice of doors to open (i.e., if your original selection happened to be correct), then he opens the *smallest number* available door. (For example, if you selected Door #1 and the car was indeed behind Door #1, then the host would always open Door #2, never Door #3.) What are the probabilities that you will win the car if you stick versus if you switch?

This Monty Crawl problem seems very similar to the original Monty Hall problem; the only difference is the host's actions when he has a *choice* of which door to open. However, the answer now is that if you see the host open the higher-numbered unselected door, then your probability of winning is 0% if you stick, and 100% if you switch. On the other hand, if the host opens the lower-numbered unselected door, then your probability of winning is 50% whether you stick or switch. Why these different probabilities? Why does the Shaky Solution not apply in this case?

3 The Proportionality Principle

To deal with these and other problems, we introduce a simple rule (see e.g. [2]; a mathematical discussion is presented in the Appendix):

The Proportionality Principle: If various alternatives are equally likely, and then some event is observed, the updated probabilities for the alternatives are proportional to the probabilities that the observed event *would* have occurred under those alternatives.

Consider some examples:

Sister in the Shower: ([2]) Either Alice or Betty is equally likely to be in the shower. Then you hear the showerer *singing*. You know that Alice always sings in the shower, while Betty only sings 1/4 of the time. What is the probability that Alice is in the shower?

Solution. By the Proportionality Principle, since Alice is four times as likely to sing as is Betty, therefore the showerer is four times as likely to be Alice as Betty. The probabilities must add up to 1, so this means that the probability is 4/5 that Alice is in the shower, and 1/5 that Betty is.

Nebulous Neighbours: Your new neighbours have two children of unknown gender. From older to younger, they are equally likely to be girl-girl, girl-boy, boy-girl, or boy-boy. One day you catch a glimpse of a child through their window, and you see that it is a girl. What is the probability that their other child is also a girl?

Solution. The probabilities that a glimpsed child will be a girl for each of the four possibilities (girl-girl, girl-boy, boy-girl, and boy-boy) are respectively 1, 1/2, 1/2, and 0. Since the probabilities must add to 1, the probabilities of these four possibilities are respectively 1/2, 1/4, 1/4, and 0. Hence, the probability is 1/2 that the other child is also a girl.

Three-Card Thriller: (e.g. [2]) A friend has three cards: one red on both sides, one black on both sides, and one red on one side and black on the other. She mixes them up in a bag, draws one at random, and places it on the table with a red side showing. What is the probability that the other side is also red?

Solution. The probability that a randomly-chosen side will be red for each of three possible cards (red-red, black-black, and red-black) are respectively 1, 0, and 1/2. So, the updated probabilities for these three possible cards must be 2/3, 0, and 1/3, respectively. That is, the probability is 2/3 that the card's other side is also red. (If you're not convinced, then consider that your friend chose one of the six possible card *sides*. Three of those sides are red, and two of the three also have a red side opposite.)

Die-controlled Coin: You roll a single six-sided die, and then flip a coin the number of times showing on the die. The coin comes up heads every time. What are the probabilities that the die showed 1, 2, 3, 4, 5, and 6, respectively?

Solution. The probabilities that the coin came up heads every time, if the die showed 1, 2, 3, 4, 5, and 6, are respectively $1/2$, $1/4$, $1/8$, $1/16$, $1/32$, and $1/64$. The probabilities for what the die showed are therefore proportional to this, namely $32/63$, $16/63$, $8/63$, $4/63$, $2/63$, and $1/63$.

4 Monty Hall Revisited

The Proportionality Principle makes the various Monty Hall variants easy. However, first a clarification is required. The original Monty Hall problem implicitly makes an **additional assumption**: if the host has a choice of which door to open (i.e., if your original selection was correct), then he is equally likely to open either non-selected door. This assumption, callously ignored by the Shaky Solution, is in fact crucial to the conclusion (as the Monty Crawl problem illustrates).

With this additional assumption, the original Monty Hall problem is solved as follows. Originally the car was equally likely to be behind Door #1 or #2 or #3, and you selected Door #1 (say). The probabilities of the host then choosing to open Door #3, when the car is actually behind Door #1, Door #2, and Door #3, are respectively $1/2$, 1 , and 0 . Hence, the updated probabilities of the car being behind each of the three doors are respectively $1/3$, $2/3$, and 0 . That is, your chance of winning the car is $1/3$ if you stick with Door #1, and $2/3$ if you switch to Door #2.

In the Monty Fall problem, suppose you select Door #1, and the host then falls against Door #3. The probabilities that Door #3 happens not to contain a car, if the car is behind Door #1, #2, and #3, are respectively 1 , 1 , and 0 . Hence, the probabilities that the car is actually behind each of these three doors are respectively $1/2$, $1/2$, and 0 . So, your probability of winning is the same whether you stick or switch.

In the Monty Crawl problem, suppose again that you select Door #1. The probabilities that the host would choose to open Door #3, if the car were behind Door #1, #2, and #3, are respectively 0 , 1 , and 0 . Hence, if the host opens Door #3, then it is certain that the car is actually behind Door #2. On the other hand, the probabilities that the host would choose to open Door #2 are respectively 1 , 0 , and 1 . Hence, if the host opens Door #2, the probabilities are now $1/2$ each that the car is behind Door #1 and Door #3.

Finally, here is a generalisation:

Monty Small Problem: In this variant, the host is only *somewhat* tired. If he has a choice of doors to open, then he has a small probability p of opening the *largest number* available door, otherwise (with probability $1-p$) he opens the *smallest number* available door. What is the probability that you will win the car if you then switch to the third door? (The case $p = 1/2$ is the original problem, while $p = 0$ is Monty Crawl.)

In this case, if you select Door #1, the probabilities that the host will open Door #3 are respectively p , 1, and 0. Hence, in this case, the probability of winning if you switch is $1/(1+p)$. [Exercise: If the host had instead opened Door #2, this probability would instead be $1/(2-p)$.]

5 Appendix: Mathematical Discussion

In mathematical terms, the Proportionality Principle says the following: If $\mathbf{P}(A_1) = \mathbf{P}(A_2) = \dots = \mathbf{P}(A_n) > 0$, and $\mathbf{P}(B) > 0$, then

$$\mathbf{P}(A_i | B) = K \mathbf{P}(B | A_i),$$

where $K > 0$ does not depend on i .

This equation is essentially a re-statement of Bayes' Theorem (e.g. [1], p. 21). Indeed,

$$\mathbf{P}(A_i | B) = \frac{\mathbf{P}(A_i \cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}(A_i) \mathbf{P}(B | A_i)}{\mathbf{P}(B)} = K \mathbf{P}(B | A_i),$$

where $K = \mathbf{P}(A_i) / \mathbf{P}(B)$, which does not depend on i since $\mathbf{P}(A_1) = \mathbf{P}(A_2) = \dots = \mathbf{P}(A_n)$.

References

- [1] Michael J. Evans and Jeffrey S. Rosenthal, *Probability and Statistics: The Science of Uncertainty*. W.H. Freeman, New York, 2003.
- [2] Jeffrey S. Rosenthal, *Struck by Lightning: The Curious World of Probabilities*. Harper-Collins Canada, 2005.
- [3] Marilyn vos Savant. *Ask Marilyn* column. Parade Magazine: Sept. 9, 1990, p. 22.
- [4] Marilyn vos Savant. *Ask Marilyn* columns. Parade Magazine: Dec. 2, 1990, p. 26; Feb. 7, 1991, p. 12; and July 7, 1991, pp. 28–29.